

Analysis of the Collatz Conjecture, Part II Constraints on Infinite Growth, Exact Block Identities, and Limits of the 3-adic Method

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Abstract

This paper records rigorous structural restrictions on positive Collatz orbits and corrects the scope of an earlier version of Part II. The description of the target set, the residue formula modulo 3, the One-Way Gate, the affine block identity, and contraction of every fixed exponent word in $\mathbb{Z}_3$ remain valid. We also give a self-contained cyclic-rotation argument showing that the divisibility condition $D \mid S(p)$ is an exact criterion for a closed admissible odd-to-odd block.

The former conclusion that these ingredients exclude infinite growth is withdrawn. The accelerated Syracuse map is a switching map: although each fixed branch is 3-adically contractive, the switching map is nowhere continuous in the induced 3-adic topology. Compactness gives pseudo-returns, but pseudo-returns occur for every fixed exponent word and do not imply an integer cycle. For a nonclosed orbit segment the cycle equation $Dn = S(p)$ is unavailable; the correct endpoint and logarithmic drift identities are stated explicitly. The basic-fractal result imported through Part I is retained as a prior result, but neither the general cycle problem nor the divergence problem is closed here. No proof of the full Collatz conjecture is claimed.

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1 Scope and corrected status

Let

$$\mathbb{N}_{>0} := \{1, 2, 3, \dots\}.$$

The standard Collatz map and the accelerated odd map are respectively

$$\text{Col}(n) = \begin{cases} 3n + 1, & n \text{ odd}, \\ n/2, & n \text{ even}, \end{cases} \quad \text{Syr}(n) = \frac{3n + 1}{2^{v_2(3n+1)}} \quad (n \text{ positive and odd}).$$

The full conjecture states that every positive integer reaches 1. Two logically distinct obstructions must be excluded:

- (i) a nontrivial positive periodic orbit;
- (ii) an unbounded positive orbit.

An exact cycle criterion addresses the first obstruction only. It does not, by itself, provide a descent theorem for nonclosed trajectories.

Status Notice 1.1. The earlier claim that the One-Way Gate, 3-adic compactness, and the divisibility condition $D \mid S(p)$ jointly exclude infinite growth is withdrawn. The step from arbitrarily deep congruent returns to an exact integer fixed-point equation is invalid. The corrected results and the remaining open target are separated below.

The basic-fractal theorem from Part I [1] is treated as an imported result and is not reproved in this paper. The purpose here is to state what survives for general finite odd-to-odd blocks and why those facts do not settle divergence.

2 The target set and elementary residue facts

Definition 2.1 (Target set). *Define*

$$\mathcal{T} := \{n \in \mathbb{N}_{>0} : 3n + 1 = 2^k \text{ for some } k \geq 1\}.$$

If an orbit reaches $n \in \mathcal{T}$, then its next accelerated odd value is 1.

Proposition 2.2 (Exact form of the target set). *The target set is*

$$\mathcal{T} = \left\{ \frac{4^m - 1}{3} : m \geq 1 \right\} = \{1, 5, 21, 85, 341, 1365, \dots\}.$$

Every member of $\mathcal{T}$ is congruent to 1 (mod 4).

Proof. The integer condition for $n = (2^k - 1)/3$ is

$$2^k \equiv 1 \pmod{3},$$

which holds exactly when $k = 2m$. Hence $n = (4^m - 1)/3$. Moreover,

$$3n = 4^m - 1 \equiv 3 \pmod{4},$$

and multiplication by $3^{-1} \equiv 3 \pmod{4}$ gives $n \equiv 1 \pmod{4}$, including $n = 1$. □

Proposition 2.3 (Residue of the next odd value). *Let n_i be positive and odd, put*

$$p_i := v_2(3n_i + 1), \quad n_{i+1} := \text{Syr}(n_i).$$

Then

$$n_{i+1} \equiv 2^{p_i} \pmod{3}.$$

Consequently, $n_{i+1} \equiv 1 \pmod{3}$ when p_i is even and $n_{i+1} \equiv 2 \pmod{3}$ when p_i is odd.

Proof. Modulo 3, the numerator $3n_i + 1$ equals 1, and $2^{-1} \equiv 2 \pmod{3}$. Therefore

$$n_{i+1} \equiv (3n_i + 1)2^{-p_i} \equiv 2^{p_i} \pmod{3}.$$

□

These facts describe necessary residue restrictions. They do not show that an arbitrary orbit must enter $\mathcal{T}$.

3 The One-Way Gate: exact scope

Lemma 3.1 (One-Way Gate). *For every positive odd integer n ,*

$$3 \nmid \text{Syr}(n).$$

Thus, after the first accelerated odd step, no odd member of a Collatz orbit is divisible by 3.

Proof. If $e = v_2(3n + 1)$, then

$$2^e \text{Syr}(n) = 3n + 1 \equiv 1 \pmod{3}.$$

The factor 2^e is invertible modulo 3, so $3 \nmid \text{Syr}(n)$. □

Remark 3.2 (What the gate does not prove). The gate confines the accelerated odd orbit after its first step to

$$\mathcal{C} := \{n \in \mathbb{N}_{>0} : n \text{ odd and } n \equiv 1, 2 \pmod{3}\}.$$

It neither forces entry into the target set $\mathcal{T}$ nor supplies a monotone quantity on $\mathcal{C}$. In particular, it does not exclude an unbounded orbit inside this corridor.

4 Exact odd-to-odd block criterion

For $q \geq 1$, define the affine branch

$$A_q(x) := \frac{3x + 1}{2^q}.$$

For a positive exponent vector

$$p = (p_1, \dots, p_a) \in \mathbb{N}_{>0}^a,$$

put

$$b := \sum_{i=1}^a p_i, \quad s_0 := 0, \quad s_k := \sum_{i=1}^k p_i,$$

and define the chronological composition

$$F_p := A_{p_a} \circ \dots \circ A_{p_1}.$$

Definition 4.1 (Admissible and closed pair). *Let $n_0 \in \mathbb{N}_{>0}$ be odd. The pair (n_0, p) is Collatz-admissible if*

$$n_{i+1} = \frac{3n_i + 1}{2^{p_{i+1}}}, \quad i = 0, \dots, a-1,$$

produces positive odd integers at every step. It is closed if $n_a = n_0$. For an admissible pair,

$$p_{i+1} = v_2(3n_i + 1)$$

holds automatically.

Proposition 4.2 (Affine composition). *Define*

$$S(p) := \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k}.$$

Then

$$F_p(x) = \frac{3^a x + S(p)}{2^b}.$$

Proof. After r steps the numerator is

$$3^r x + \sum_{k=0}^{r-1} 3^{r-1-k} 2^{s_k}.$$

Applying $A_{p_{r+1}}$ multiplies the existing numerator by 3, adds 2^{s_r} , and changes the denominator to 2^{s_r+1} . Induction gives the formula. $\square$

Put

$$D := 2^b - 3^a.$$

As a formal affine map, F_p has a positive real fixed point exactly when $D > 0$, in which case the fixed point is $S(p)/D$. Formal integrality alone must still be related to the prescribed odd steps.

Definition 4.3 (Cyclic rotation). *For $p = (p_1, p_2, \dots, p_a)$, let*

$$\rho p := (p_2, \dots, p_a, p_1).$$

Write $\rho^i p$ for the i -fold cyclic rotation.

Lemma 4.4 (Cyclic-rotation identity). *For every positive exponent vector p ,*

$$3S(p) + D = 2^{p_1} S(\rho p).$$

Proof. Using $s_1 = p_1$,

$$\begin{aligned} 3S(p) + D &= 3 \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k} + 2^b - 3^a \\ &= \sum_{k=1}^{a-1} 3^{a-k} 2^{s_k} + 2^b \\ &= 2^{p_1} \left(\sum_{k=1}^{a-1} 3^{a-k} 2^{s_k - p_1} + 2^{b-p_1} \right). \end{aligned}$$

The expression in parentheses is $S(\rho p)$. $\square$

Theorem 4.5 (Exact odd-to-odd cycle criterion). *For $p \in \mathbb{N}_{>0}^a$, the following statements are equivalent:*

- (i) *there exists $n_0 \in \mathbb{N}_{>0}$ for which (n_0, p) is Collatz-admissible and closed;*
- (ii) *$D = 2^b - 3^a > 0$ and $D \mid S(p)$.*

When these conditions hold, the odd members of the orbit are

$$n_i = \frac{S(\rho^i p)}{D}, \quad i = 0, \dots, a-1.$$

Proof. For a closed admissible pair, Proposition 4.2 gives $Dn_0 = S(p)$, and positivity gives $D > 0$.

Conversely, assume $D > 0$ and $D \mid S(p)$. The number D is odd, so Lemma 4.4 implies $D \mid S(\rho p)$, and iteration gives divisibility for every rotation. Each rotated sum is odd: its first term is 3^{a-1} , while all later terms are even. Hence every $n_i = S(\rho^i p)/D$ is a positive odd integer. Applying the rotation identity to $\rho^i p$ and dividing by D yields

$$3n_i + 1 = 2^{p_{i+1}} n_{i+1}.$$

Since n_{i+1} is odd, the valuation is exactly p_{i+1} . Finally, $\rho^a p = p$, so the orbit closes. $\square$

Remark 4.6 (Cycle arithmetic is not a divergence argument). Theorem 4.5 is an exact criterion for a closed block. For a nonclosed segment $n_a \neq n_0$, its left-hand side is not Dn_0 . Therefore the failure of $D \mid S(p)$ for a given word does not prevent that word from occurring along a nonclosed or unbounded trajectory. The special basic-fractal cycle arithmetic remains part of Part I; divergence requires a separate descent argument.

Part I also develops neighboring-quotient, finite-window, and prime-power certificates for closed words in the $\{1, 2\}$ class. Those certificates target the closed-word divisibility condition and are unavailable for a nonclosed orbit segment; consequently, they do not resolve the uniform descent problem studied here.

5 What the 3-adic method proves

For each fixed $e \geq 1$, the formula

$$A_e(x) = \frac{3x + 1}{2^e}$$

defines an affine map on all of $\mathbb{Z}_3$, because 2 is a 3-adic unit.

Proposition 5.1 (Fixed-branch and fixed-word contraction). *For all $x, y \in \mathbb{Z}_3$,*

$$|A_e(x) - A_e(y)|_3 = \frac{1}{3}|x - y|_3.$$

More generally, for a fixed word $p \in \mathbb{N}_{>0}^a$,

$$|F_p(x) - F_p(y)|_3 = 3^{-a}|x - y|_3.$$

The map F_p has the unique 3-adic fixed point

$$\xi_p = \frac{S(p)}{D} \in \mathbb{Z}_3,$$

and $F_p^r(x) \rightarrow \xi_p$ in $\mathbb{Z}_3$ for every $x \in \mathbb{Z}_3$.

Proof. The difference of two branch values is

$$A_e(x) - A_e(y) = \frac{3(x - y)}{2^e}.$$

For a word of length a , Proposition 4.2 gives slope $3^a/2^b$, whose 3-adic norm is 3^{-a} . Moreover, $D = 2^b - 3^a \not\equiv 0 \pmod{3}$, so D is a unit in $\mathbb{Z}_3$. The fixed-point and convergence statements follow from the contraction mapping theorem. $\square$

Status Notice 5.2. Repeating one preassigned affine word is not the same operation as iterating Syr . The exponent $v_2(3n + 1)$ changes with the input, and the resulting switching map is not globally 3-adically contractive.

For $e \geq 1$, define the exponent stratum

$$\Omega_e := \{n \in \mathbb{N}_{>0} : n \text{ odd and } v_2(3n+1) = e\}.$$

Lemma 5.3 (Dense exponent strata). *Every Ω_e is dense in $\mathbb{Z}_3$.*

Proof. The exact valuation condition is equivalent to

$$n \equiv 3^{-1}(2^e - 1) \pmod{2^{e+1}},$$

where the inverse is taken modulo 2^{e+1} . Given any 3-adic ball $n \equiv c \pmod{3^M}$, the Chinese remainder theorem combines this condition with the prescribed residue modulo 3^M and gives a positive solution. $\square$

Theorem 5.4 (Nowhere continuity of the switching map). *On the positive odd integers with the metric induced from $\mathbb{Z}_3$, the accelerated map Syr is discontinuous at every point. Consequently, it has no continuous extension to $\mathbb{Z}_3$.*

Proof. Fix a positive odd n and let $e_0 = v_2(3n+1)$. Choose e of parity opposite to e_0 . Lemma 5.3 gives positive odd integers $y_M \rightarrow n$ with $v_2(3y_M+1) = e$. For every odd z ,

$$\text{Syr}(z) \equiv 2^{-v_2(3z+1)} \pmod{3}.$$

The two exponent parities are different, so $\text{Syr}(y_M) \not\equiv \text{Syr}(n) \pmod{3}$. Their 3-adic distance remains equal to 1, proving discontinuity. $\square$

Remark 5.5 (Explicit discontinuity certificate). For

$$x_M = 1 + 2 \cdot 3^M,$$

we have $x_M \rightarrow 1$ in $\mathbb{Z}_3$, but

$$\text{Syr}(x_M) = 2 + 3^{M+1} \rightarrow 2 \neq 1 = \text{Syr}(1).$$

Remark 5.6 (Correct compactness consequence). Compactness of $\mathbb{Z}_3$ guarantees that every infinite sequence of positive integers has a 3-adically convergent subsequence. For an accelerated Collatz orbit, all terms after the first are 3-adic units, that is, they lie in $\mathbb{Z}_3^\times = \{x \in \mathbb{Z}_3 : |x|_3 = 1\}$, by Lemma 3.1. Compactness does not imply convergence of the full orbit, uniqueness of an accumulation point, or an exact return in $\mathbb{N}_{>0}$.

6 Admissible words and arbitrarily deep pseudo-returns

Lemma 6.1 (Admissible residue class of a fixed word). *Let $p \in \mathbb{N}_{>0}^a$ and $b = \sum_i p_i$. There is a unique odd residue $r_p \pmod{2^{b+1}}$ such that a positive integer n realizes the exact exponent word p if and only if*

$$n \equiv r_p \pmod{2^{b+1}}.$$

It is characterized by

$$3^a r_p + S(p) \equiv 2^b \pmod{2^{b+1}},$$

or equivalently

$$r_p \equiv 3^{-a}(2^b - S(p)) \pmod{2^{b+1}}.$$

Proof. For $1 \leq j \leq a$, put

$$S_j := \sum_{k=0}^{j-1} 3^{j-1-k} 2^{s_k}, \quad N_j := 3^j n + S_j.$$

The formal endpoint after j steps is $N_j/2^{s_j}$. Requiring the terminal endpoint to be odd is exactly

$$N_a \equiv 2^b \pmod{2^{b+1}},$$

which determines one odd residue for n , since 3^a is invertible modulo 2^{b+1} .

For $j < a$, reduce the terminal congruence modulo 2^{s_j+1} . It gives

$$3^{a-1-j}(3N_j + 2^{s_j}) \equiv 0 \pmod{2^{s_j+1}}.$$

The odd factor is invertible. Hence $2^{s_j} \mid N_j$ and $N_j/2^{s_j}$ is odd. Thus all intermediate divisions are exact and have the prescribed valuations. $\square$

Theorem 6.2 (CRT pseudo-returns for every fixed word). *Let p be any fixed positive exponent word and $M \geq 1$. There are infinitely many positive odd integers n that realize p exactly and satisfy*

$$F_p(n) \equiv n \pmod{3^M}.$$

Proof. Combine the congruences

$$n \equiv r_p \pmod{2^{b+1}}, \quad Dn \equiv S(p) \pmod{3^M}.$$

The second has a unique solution modulo 3^M , since D is a 3-adic unit. The moduli are coprime, so the Chinese remainder theorem gives one class modulo $2^{b+1}3^M$ with infinitely many positive representatives. Lemma 6.1 gives the exact itinerary, and

$$F_p(n) - n = \frac{S(p) - Dn}{2^b} \equiv 0 \pmod{3^M}.$$

$\square$

Proposition 6.3 (Nondivisible words with pseudo-returns at every depth). *Let*

$$q = (1, 2, 2, 1, 2, 2), \quad p = q^r,$$

and put

$$G_r := \sum_{\ell=0}^{r-1} 729^{r-1-\ell} 1024^\ell.$$

Then

$$D(p) = 295G_r > 0, \quad S(p) = 1357G_r,$$

so $D(p) \nmid S(p)$ for every $r \geq 1$. Nevertheless, for every $r, M \geq 1$, Theorem 6.2 gives infinitely many genuine length-6r Collatz blocks whose endpoints are congruent modulo 3^M .

Proof. For one block,

$$F_q(x) = \frac{729x + 1357}{1024}.$$

Iterating this affine map r times gives the displayed formulas. Divisibility would require $295 \mid 1357$, which is false. The pseudo-return assertion follows from Theorem 6.2. $\square$

Remark 6.4 (A concrete pseudo-return). For $p = (1, 1, 2, 2)$,

$$F_p(x) = \frac{81x + 73}{64}, \quad D = -17 \nmid 73 = S(p).$$

The admissible start $n = 65863$ gives

$$65863 \mapsto 98795 \mapsto 148193 \mapsto 111145 \mapsto 83359$$

with exponent word $(1, 1, 2, 2)$, and

$$83359 - 65863 = 17496 = 24 \cdot 3^6.$$

Thus even a genuine deep 3-adic pseudo-return need not be an exact cycle.

7 Exact finite-segment identities and the open descent problem

Let

$$n_0, n_1, \dots, n_a$$

be a finite accelerated odd Collatz segment, with

$$p_{i+1} = v_2(3n_i + 1), \quad b_a := \sum_{i=1}^a p_i,$$

and let $S_a = S(p_1, \dots, p_a)$.

Proposition 7.1 (Exact endpoint identity). *Every finite accelerated odd segment satisfies*

$$2^{b_a} n_a - 3^a n_0 = S_a.$$

Equivalently,

$$(2^{b_a} - 3^a) n_0 = S_a - 2^{b_a} (n_a - n_0).$$

The cycle equation $Dn_0 = S_a$ is obtained only when $n_a = n_0$.

Proof. Apply Proposition 4.2 to the realized exponent word and multiply its endpoint equation by 2^{b_a} . The second formula is a rearrangement. $\square$

Proposition 7.2 (Exact logarithmic drift). *Every finite accelerated odd segment satisfies*

$$\log \frac{n_a}{n_0} = a \log 3 - b_a \log 2 + \sum_{i=0}^{a-1} \log \left(1 + \frac{1}{3n_i} \right).$$

Proof. For each step,

$$\frac{n_{i+1}}{n_i} = \frac{3}{2^{p_{i+1}}} \left(1 + \frac{1}{3n_i} \right).$$

Taking logarithms and summing proves the identity. $\square$

Open Problem 7.3 (Uniform descent target). *Prove that for every positive odd $n_0 > 1$, its accelerated orbit has a finite prefix with $n_a < n_0$. By Proposition 7.2, this is equivalent to proving, for some $a \geq 1$,*

$$b_a \log 2 - a \log 3 > \sum_{i=0}^{a-1} \log \left(1 + \frac{1}{3n_i} \right).$$

A proof of Problem 7.3 would yield convergence by strong induction on the starting value. None of the One-Way Gate, fixed-word contraction, compactness, or cycle divisibility establishes this inequality for every orbit. The divergence question therefore remains open. Lagarias gives a broad overview of the $3x + 1$ problem and its known restrictions [2]. Tao's almost-all result is a major metric advance, but it does not give the required descent statement for every starting value [3].

8 The only positive integer one-step fixed point

Lemma 8.1 (Narrow one-step fixed-point statement). *Let $n \in \mathbb{N}_{>0}$ be odd. If $\text{Syr}(n) = n$, then $n = 1$.*

Proof. Put $k = v_2(3n + 1) \geq 1$. From $\text{Syr}(n) = n$,

$$3n + 1 = 2^k n, \quad n(2^k - 3) = 1.$$

Both factors are integers. Positivity forces

$$n = 1, \quad 2^k - 3 = 1,$$

so $k = 2$. □

Remark 8.2. This lemma classifies one-step positive integer fixed points only. It does not exclude longer cycles, prove convergence of a hypothetical orbit in $\mathbb{Z}_3$, or imply that a sequence converging 3-adically to 1 must ever equal 1 in $\mathbb{N}_{>0}$.

9 Conclusions

The corrected Part II establishes the following rigorous facts:

- (i) the exact form and elementary residue properties of the target set $\mathcal{T}$;
- (ii) the One-Way Gate for the accelerated odd map;
- (iii) the exact affine and cyclic-rotation criterion for a closed admissible exponent word;
- (iv) contraction of each fixed 3-adic branch and fixed word, together with nowhere continuity of the switching map;
- (v) the existence of arbitrarily deep admissible pseudo-returns even for words that cannot close;
- (vi) the exact endpoint and logarithmic drift identities for every finite odd segment.

These results impose genuine structural constraints, but they do not exclude unbounded Collatz trajectories. Exact cycles and divergence are different problems: cycle arithmetic belongs to the closed-word criterion and the basic-fractal work of Part I, whereas divergence requires a uniform descent mechanism such as Problem 7.3. The strengthened neighboring-quotient and prime-power certificates in Part I do not change this separation. The full Collatz conjecture remains open.

References

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